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Twisted Six Dimensional Gauge Theories on Tori, Matrix Models, and Integrable Systems

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Abstract

We use the Dijkgraaf-Vafa technique to study massive vacua of 6D SU(N) SYM theories on tori with R-symmetry twists. One finds a matrix model living on the compactification torus with a genus 2 spectral curve. The Jacobian of this curve is closely related to a twisted four torus T in which the Seiberg-Witten curves of the theory are embedded. We also analyze R-symmetry twists in a bundle with nontrivial first Chern class which yields intrinsically 6D SUSY breaking and a novel matrix integral whose eigenvalues float in a sea of background charge. Next we analyze the underlying integrable system of the theory, whose phase space we show to be a system of N-1 points on T. We write down an explicit set of Poisson commuting Hamiltonians for this system for arbitrary N and use them to prove that equilibrium configurations with respect to all Hamiltonians correspond to points in moduli space where the Seiberg-Witten curve maximally degenerates to genus 2, thereby recovering the matrix model spectral curve. We also write down a conjecture for a dual set of Poisson commuting variables which could shed light on a particle-like interpretation of the system.

1 Introduction

Recent advances in the study of $\mathcal{N} = 1$ supersymmetric gauge theories [1]-[4] have provided [5]-[10] a unifying framework for deriving exact results for superpotentials of $\mathcal{N} = 1$ theories [11]-[14] and moduli spaces of $\mathcal{N} = 2$ theories [15]-[16].

One of the remarkable aspects of the Dijkgraaf-Vafa (DV) technique [1] is that it can also be applied to nonrenormalizable $\mathcal{N} = 1$ theories. For such theories, there still exists a subset of questions with finite answers that are independent of the UV regularization. These are the questions pertaining to the chiral ring and the DV technique provides a solution to those questions.

This observation allows one to study higher dimensional gauge theories [17]-[18]. The purpose of this paper is to explore the application of the DV technique to 5+1D Super-Yang-Mills theories and to relate this approach to the viewpoint of integrable systems. The DV technique applies most directly to 3+1D gauge theories with $\mathcal{N} = 2$ or $\mathcal{N} = 1$ supersymmetry. We will therefore compactify the 5+1D gauge theories on T^2 to 3+1D. A simple way to break supersymmetry to $\mathcal{N} = 2$ is to introduce R-symmetry twists. The 5+1D theory has $SU(2) \times SU(2)$ R-symmetry which can be broken down to $SU(2) \times U(1)$ with appropriate boundary conditions (i.e. R-symmetry Wilson lines).

The resulting 3+1D low-energy effective action was studied in [19]-[20]. The effective action is described in terms of Seiberg-Witten curves that can be embedded inside a fixed T^4. The complex structure of the T^4 should be determined from the complex structure of the physical T^2, the ratio between its area and the 5+1D Yang-Mills coupling constant squared, and the values of the R-symmetry twists.

One can also compactify on T^2 with more complicated R-symmetry boundary conditions so as to preserve only $\mathcal{N} = 1$ in 3+1D. One way to do that was studied in [21]. For this purpose a $U(1)$ subgroup of the R-symmetry group is picked and then twisted so as to form a $U(1)$ bundle with nonzero first Chern class over T^2.

In this paper we will study these resulting 3+1D theories from the perspective of the DV conjecture and integrable systems. The paper is organized as follows. In section (2) we explain the setup starting from 5+1D and compactifying down to 3+1D. In section (3) we write down and solve the matrix model required to obtain the degenerated genus 2 Seiberg-Witten curves of the compactified theories in their massive vacua. In section (4) we show how putting R-symmetry twists in a bundle of nontrivial Chern class over the compactification torus yields a modified matrix integral that could explain an intrinsically six dimensional mechanism for breaking SUSY to $\mathcal{N} = 1$. We move on to the integrable systems approach in section (5). Here we review the various relations between extremization of DV superpotentials, degenerating Seiberg-Witten curves, and equilibrium configurations of integrable systems. We then use algebraic-geometry

techniques (a sort of Fourier-Mukai transform) to reformulate the integrable system associated to the 6D theory in terms of a set of points on T^4. This technique also yields an explicit set of Poisson commuting Hamiltonians for this system, written in terms of theta functions on T^4. We prove that configurations that are at equilibrium with respect to all such Hamiltonian flows imply a degeneration of the Seiberg-Witten curve to genus 2. In section 6 we write down a conjecture for an alternate set of Poisson commuting observables that could yield physical insight into the nature of the integrable system. We end in section 7 with conclusions and directions for future work. Also, for the convenience of the reader we assemble facts about elliptic functions on T^2 in appendix A and in appendix B we review higher dimensional theta functions, Jacobians of Riemann surfaces and the Abel-Jacobi map.

Some of the earlier results in this paper appeared independently in [22], where the matrix model was used to extract Seiberg-Witten curves. Also, upon completion of this paper, [23] appeared which also discusses the integrable systems viewpoint.

2 The setup: 5+1D gauge theories on T^2

Our starting point is 5+1D Super-Yang-Mills theory with gauge group $SU(N)$ and with $\mathcal{N} = (1,1)$ supersymmetry. The coupling constant g_{YM} has dimensions of length. We denote the coordinates by $x_0 \dots x_5$ and we set

$$z \stackrel{\text{def}}{=} x_4 + ix_5, \qquad \overline{z} \stackrel{\text{def}}{=} x_4 - ix_5.$$

The R-symmetry is $Spin(4) = SU(2) \times SU(2)$. Our indexing notation is as follows

Symbol	Range	Representation
$\mu, \nu, \cdots$	$0 \dots 5$	Spacetime vector
$a, b, \cdots$	$1, 2$	R-symmetry "left" spinor
$\dot{a}, \dot{b}, \cdots$	$\dot{1}, \dot{2}$	R-symmetry "right" spinor
$\alpha, \beta, \cdots$	$1, \dots, 4$	spacetime left Weyl spinor
$\dot{\alpha}, \dot{\beta}, \cdots$	$\dot{1}, \dots, \dot{4}$	spacetime right Weyl spinor

The field contents of the theory is described in the following table.

Symbol	Field	$SU(2) \times SU(2)$ representation
A_μ	Gauge	$(\mathbf{1}, \mathbf{1})$
$\Phi^{\alpha\dot{\beta}}$	Scalar	$(\mathbf{2}, \mathbf{2})$
$\psi^{a\alpha}$	Left moving fermion	$(\mathbf{2}, \mathbf{1})$
$\psi^{\dot{a}\dot{\alpha}}$	Right moving fermion	$(\mathbf{1}, \mathbf{2})$

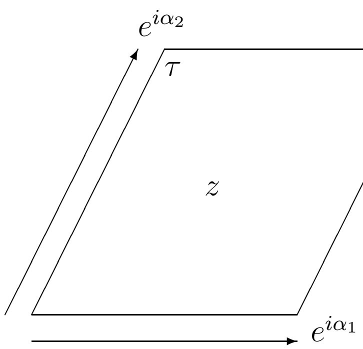

Figure 1: The boundary conditions along the T^2 in the compact directions $4-5$. The scalar and fermion fields $\Phi^{\alpha\dot{\alpha}}$ and $\psi^{\dot{a}\dot{\alpha}}$ pick up phases when translated along cycles of T^2.

The Lagrangian is

$$\mathcal{L} = \frac{1}{{g_{\mathrm{YM}}}^2}\mathrm{tr}\left\{\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\Phi_{\alpha\dot{\alpha}}D^\mu\Phi^{\beta\dot{\beta}} - \frac{1}{2}[\Phi^{\alpha\dot{\alpha}},\Phi^{\beta\dot{\beta}}][\Phi_{\alpha\dot{\alpha}},\Phi_{\beta\dot{\beta}}] + \text{fermions}\right\}.$$

2.1 Compactification on T^2 with twists

Let us now compactify on a torus Σ_τ with complex structure τ and area A. We choose the complex coordinate z such that

$$z \sim z+1 \sim z+\tau.$$

We can use one of the two $SU(2)$ factors of the R-symmetry to twist the compactification. For this purpose we choose a $U(1) \subset SU(2)$ of the second factor (corresponding to the dotted $\dot{\alpha},\dot{\beta}$ indices). We then pick two constant elements in this $U(1)$ subgroup which we represent as $e^{i\alpha_1}, e^{i\alpha_2}$. Here α_1,α_2 are constant phases corresponding to boundary conditions in the $z \sim z+1$ and $z \sim z+\tau$ directions, respectively. Explicitly, we set the boundary conditions

$$\begin{aligned} \Phi^{\alpha\dot{1}}(z) &= e^{i\alpha_1}\Phi^{\alpha\dot{1}}(z+1) = e^{i\alpha_2}\Phi^{\alpha\dot{1}}(z+\tau), \qquad \Phi^{\alpha\dot{2}}(z) = e^{-i\alpha_1}\Phi^{\alpha\dot{2}}(z+1) = e^{-i\alpha_2}\Phi^{\alpha\dot{2}}(z+\tau), \\ \psi^{\dot{a}\dot{1}}(z) &= e^{i\alpha_1}\psi^{\dot{a}\dot{1}}(z+1) = e^{i\alpha_2}\psi^{\dot{a}\dot{1}}(z+\tau), \qquad \psi^{\dot{a}\dot{2}}(z) = e^{-i\alpha_1}\psi^{\dot{a}\dot{2}}(z+1) = e^{-i\alpha_2}\psi^{\dot{a}\dot{2}}(z+\tau). \end{aligned} \tag{1}$$

The remaining fields have periodic boundary conditions.

For nonzero phases α_1,α_2 this compactification breaks half of the supersymmetry and preserves $\mathcal{N}=2$ in the remaining noncompact 3+1D. The fields in (1) become massive and classically we are left with a massless $\mathcal{N}=2$ vector multiplet in 3+1D.

2.2 Compactification on T^2 with with nonzero c_1

We can break the supersymmetry down to $\mathcal{N} = 1$ in 3+1D by introducing a nonzero Chern class for the $U(1)$-bundle over T^2 [21]. As before we fix a $U(1) \subset SU(2)$ subgroup of the second $SU(2)$ factor of the R-symmetry group. We then pick a nondynamical $U(1)$ gauge field $A^{(\text{n.d.})}$ with components $A^{(\text{n.d.})}{}_z(z, \overline{z}), A^{(\text{n.d.})}{}_{\overline{z}}(z, \overline{z})$ only along T^2 and with constant field-strength such that $\int_{T^2} dA^{(\text{n.d.})} = c_1 = n$.

We then take periodic boundary conditions for all the fields but modify the covariant derivatives of the fields that are charged under $U(1)$ [i.e. the fields appearing in (1)] to include the nondynamical gauge field $A^{(\text{n.d.})}$.

As explained in [21], such a modification to the Lagrangian affects the fermions and scalars differently but it is possible to preserve partial supersymmetry by adding an explicit coupling to one of the components of the D-terms of the theory. The extra coupling is of the form

$$\int \partial_{[z} A^{(\text{n.d.})}{}_{\overline{z}]} D^{(3)} \, d^2 z, \qquad D^{(3)} \equiv \Phi^{1\dot{1}}\Phi^{2\dot{2}} + \Phi^{1\dot{2}}\Phi^{2\dot{1}}.$$

Here $D^{(3)}$ is a member of an $SU(2)_R$ triplet. This term explicitly breaks the R-symmetry down to $U(1)$, and the unbroken supersymmetry is indeed just $\mathcal{N} = 1$.

2.3 Quiver theories

We can generalize the discussion above to quiver theories. These are chiral gauge theories with $\mathcal{N} = (1, 0)$ in 5+1D. The gauge group is

$$G = SU(N)_1 \times SU(N)_2 \times \cdots \times SU(N)_k$$

where the subscript is a label of the factor. The R-symmetry is $SU(2)_R$ and there is an additional global $U(1)$ flavor symmetry. The fields of the theory fall into vector multiplet and hypermultiplet representations of $\mathcal{N} = (1, 0)$ supersymmetry. The vector multiplet is in the adjoint representation of the gauge group and its field content is described in the following table.

Symbol	Field	$SU(2)_R$ representation	$U(1)$ charge
A_μ	Gauge	$\mathbf{1}$	0
$\chi^{a\alpha}$	Left moving gluino	$\mathbf{2}$	0

There are k hypermultiplets. The j^{th} ($j = 1 \dots k$) hypermultiplet has fields in the product of the anti-fundamental representation $\overline{N}$ of $SU(N)_j$ and the fundamental representation N of $S(N)_{j+1}$ (with the cyclic convention $k + 1 \equiv 1$). Its field contents is described in the following table.

Symbol	Field	$SU(2)_R$ representation	$U(1)$ charge
$\Phi^{\alpha}_{j,j+1}$	complex scalar	$\mathbf{2}$	$+1$
$\psi^{\dot{a}}_{j,j+1}$	right moving spinor	$\mathbf{1}$	$+1$

We can proceed to compactify these theories on T^2 either with $\mathcal{N}=2$ preserving $U(1)$ twists as in (2.1) or with a $\mathcal{N}=1$ preserving nontrivial $U(1)$ bundle with first Chern class $c_1 = n$ as in (2.2). In the latter case, we obtain n generations of chiral matter in 3+1D [21].

3 $\mathcal{N}=1$ Massive vacua from DV matrix integrals

We will now use the DV technique to probe the massive vacua of $\mathcal{N}=1$ deformations of the twisted 3+1D $\mathcal{N}=2$ theory described in section (2.1). We use the same notation as in section (2.1) but for convenience take the area A of Σ_τ to be 1, which just sets the effective bare 3+1D gauge coupling to be the same as the 5+1D coupling. The starting point for implementing the DV technique is to write the Lagrangian of the 5+1D theory in 3+1D $\mathcal{N}=1$ superspace. The twists $e^{i\alpha_1}, e^{i\alpha_2}$ are in a global $U(1)$ group. The 5+1D vector field decomposes into a 3+1D vector field, that is part of a vector multiplet $\mathcal{V}$, and a 3+1D complex scalar field $A_{\overline{z}} \equiv A_4 - iA_5$ that is the scalar component of a chiral multiplet $\mathcal{A}_{\overline{z}}$. The 5+1D scalars fall into two 3+1D chiral multiplets Φ_+, Φ_- with opposite $U(1)$ charges. All the fields are functions of $z, \overline{z}$ and are summarized, together with their global $U(1)$ charges, in the following table:

Symbol	5+1D Field	Multiplet	$U(1)$ charge
$\mathcal{V}$	Gauge	vector	0
$\mathcal{A}_{\overline{z}}$	Gauge	chiral	0
Φ_+	Hyper	chiral	$+1$
Φ_-	Hyper	chiral	-1

The DV technique states that the properties of the chiral ring of the effective 3+1D theory above can be deduced from the path integral of the internal two dimensional "holomorphic" bosonic gauge theory,

$$\mathcal{Z} = \int [D\mathcal{A}_{\overline{z}}][D\Phi_+][D\Phi_-] e^{-\frac{1}{g_{\rm st}{}^2} W} \tag{2}$$

where the superpotential is

$$W = \int_{\Sigma_\tau} d^2 z \,\mathrm{tr}\,\{\Phi_+ \partial_z \Phi_- - i\Phi_+[\mathcal{A}_{\overline{z}}, \Phi_-] + W[\mathcal{A}_{\overline{z}}]\}. \tag{3}$$

Here $W[A_{\overline{z}}]$, which we insert by hand to break supersymmetry down to $\mathcal{N}=1$, can be an arbitrary gauge invariant holomorphic function of the chiral field $\mathcal{A}_{\overline{z}}$. This superpotential is the dimensional reduction down to 2D of the holomorphic Chern-Simons action in 6D that appears when the DV technique is applied

to 10D Super-Yang-Mills theory [17]. All fields are $\hat{N} \times \hat{N}$ matrices where $\hat{N}$ has no relation to the physical N of $SU(N)$. The fields Φ_+ and Φ_- are required to have boundary conditions

$$\Phi_\pm(z) = e^{\pm i\alpha_1}\Phi_\pm(z+1) = e^{\pm i\alpha_2}\Phi_\pm(z+\tau). \tag{4}$$

We must now specify the path in field space over which the path integral in (2) should be performed. Recall that in the 3+1D DV technique the path integral is of the form

$$\int [D\Phi] e^{-\frac{1}{g^2}W(\Phi)} \longrightarrow \int \prod_{j=1}^{\hat{N}} d\lambda_i \prod_{i<j}(\lambda_i - \lambda_j)^2 e^{-\frac{1}{g^2}\sum_i W(\lambda_i)}.$$

where $\lambda_1, \dots, \lambda_N$ are the complex eigenvalues of Φ. Note that the integral is performed only over real λ_i's (or any other suitably chosen path in the complex λ_i-plane). In other words, the measure is $\prod_i d\lambda_i$ and not $\prod_i d^2\lambda_i$. Thus, the integration over $[D\Phi]$ is performed not over the entire Φ-space but only over the subspace restricted by $\Phi^\dagger = \Phi$. Similarly in the 5+1D case, one can choose the real slice $\Phi_+^\dagger = \Phi_-$. However in the integral over the antiholomorphic connection $\mathcal{A}_{\overline{z}}$, the only gauge invariant data to be integrated over are the holonomies $\hat{W}_1$ and $\hat{W}_2$ of $\mathcal{A}_{\overline{z}}$ around the two one cycles of Σ_τ. These are defined as

$$\hat{W}_1 \stackrel{\text{def}}{=} e^{i\oint_{z\to z+1}\mathcal{A}_{\overline{z}}d\overline{z}}, \qquad \hat{W}_2 \stackrel{\text{def}}{=} e^{i\oint_{z\to z+\tau}\mathcal{A}_{\overline{z}}d\overline{z}}, \qquad \hat{W}_1, \hat{W}_2 \in U(N), \qquad \hat{W}_1\hat{W}_2 = \hat{W}_2\hat{W}_1.$$

By gauge transformations, we can simultaneously diagonalize $\hat{W}_1, \hat{W}_2$

$$\hat{W}_1 = \begin{pmatrix} e^{2\pi i\lambda_1^{(1)}} & & & \\ & e^{2\pi i\lambda_1^{(2)}} & & \\ & & \ddots & \\ & & & e^{2\pi i\lambda_1^{(N)}} \end{pmatrix}, \qquad \hat{W}_2 = \begin{pmatrix} e^{2\pi i\lambda_2^{(1)}} & & & \\ & e^{2\pi i\lambda_2^{(2)}} & & \\ & & \ddots & \\ & & & e^{2\pi i\lambda_2^{(N)}} \end{pmatrix}, \tag{5}$$

and then combine the two sets of eigenvalues into one set of complex variables

$$\lambda_j \stackrel{\text{def}}{=} \lambda_2^{(j)} + \lambda_1^{(j)}\tau, \qquad j = 1 \dots \hat{N}. \tag{6}$$

The periodicity of the phases $\lambda_1^{(j)}, \lambda_2^{(j)}$ implies that λ_j is naturally defined on a (dual) T^2 of complex structure τ,

$$\lambda_j \sim \lambda_j + 1 \sim \lambda_j + \tau.$$

So the path integral over $\mathcal{A}_{\overline{z}}$ reduces to a finite dimensional integral over $\lambda_1, ..\lambda_{\hat{N}}$, but the change of variables incurs a nontrivial Jacobian factor. Since each integration variable λ_i naturally takes values on a torus, one can view this finite dimensional integral simply as a compactified Hermitian matrix integral. As is well known in compactifying Matrix theory on tori, one thinks of each eigenvalue λ_i as living on the

complex plane; then adding in all images $\lambda_i + m + n\tau$, with $m, n \in \mathbb{Z}$, effectively compactifies this plane. The Jacobian of the change of measure is then simply

$$\prod_{i<j} \prod_{m,n} (\lambda_i - \lambda_j + m + n\tau)^2 \tag{7}$$

which is similar to the Vandermonde determinant appearing in the usual Hermitian matrix integrals, but with contributions from differences of eigenvalue images not related by $z \sim z + 1 \sim z + \tau$. Furthermore, gauge invariance dictates that our added superpotential W is an elliptic function in λ_i.

Upon performing the change of variables (5) and (6), and taking into the account the Jacobian factor (7), the path integral (2) becomes

$$\mathcal{Z} = \int d\lambda_1 \ldots d\lambda_{\hat{N}} \prod_{i<j} \prod_{m,n} (\lambda_i - \lambda_j + m + n\tau)^2 [D\Phi_+][D\Phi_-] \exp\left[-\frac{1}{{g_{\rm st}}^2} \int_{T^2} \operatorname{tr} \Phi_- D_{\overline{z}} \Phi_+ + \sum_i W(\lambda_i)\right].$$

The integral over $[D\Phi_+][D\Phi_-]$ is gaussian and reduces to calculating the functional determinant $\det[D_{\overline{z}}]$. This can be done by explicitly calculating the eigenvalues of $\partial_{\overline{z}}$ in the space of functions on the dual Σ_τ obeying specific boundary conditions. For the ijth element of Φ_+ these boundary conditions correspond to picking up a phase $\exp i[\lambda_1^{(i)} - \lambda_1^{(j)} + \alpha_1]$ under $z \to z + 1$ and $\exp i[\lambda_2^{(i)} - \lambda_2^{(j)} + \alpha_2]$ under $z \to z + \tau$. In this space of functions the product of eigenvalues of $\partial_{\overline{z}}$ can be written as

$$\prod_{m,n} (\lambda_i - \lambda_j + \alpha + m + n\tau), \qquad m, n \in \mathbb{Z}.$$

Here we have defined the new complex twist parameter

$$\alpha \equiv \alpha_2 + \alpha_1 \tau,$$

and the λ_i are defined in (6). Thus integrating out Φ_+, Φ_- leaves us with the integral

$$\mathcal{Z} = \int d\lambda_1 \ldots d\lambda_{\hat{N}} \prod_{i<j} \frac{\prod_{m,n}(\lambda_i - \lambda_j + m + n\tau)^2}{\prod_{m,n}(\lambda_i - \lambda_j + \alpha + m + n\tau)(\lambda_i - \lambda_j - \alpha + m + n\tau)} e^{-\frac{1}{{g_{\rm st}}^2} \sum_i W(\lambda_i)}$$

In the denominator, corresponding to the functional determinant, we split the product over $i \neq j$ into a product over $i < j$ and $i > j$ and relabeled the second product. The measure is now an elliptic function on T^2 with a double zero at $\lambda_i - \lambda_j = 0$, and single poles at $\lambda_i - \lambda_j \pm \alpha = 0$. This data determines the measure uniquely, up to a multiplicative constant. Recall that the theta function $\vartheta_1(z)$ has a simple zero at $z = 0$, and that functions of the form

$$\prod_i \frac{\vartheta_1(z - a_i)}{\vartheta_1(z - b_i)}$$

are doubly periodic if $\sum_i a_i \equiv \sum_i b_i \, (\mathrm{mod}\, \mathbb{Z})$. Then we can rewrite the above integral as

$$\mathcal{Z} = \int d\lambda_1 \dots d\lambda_{\hat{N}} \prod_{i<j} \frac{\vartheta_1(\lambda_i - \lambda_j)^2}{\vartheta_1(\lambda_i - \lambda_j + \alpha)\vartheta_1(\lambda_i - \lambda_j - \alpha)} e^{-\frac{1}{g_{\mathrm{st}}{}^2}\sum_i W(\lambda_i)} \tag{8}$$

$$\equiv \int d\lambda_1 \dots d\lambda_{\hat{N}} e^{-\sum_i \mathcal{S}(\lambda_i)}, \tag{9}$$

where

$$\mathcal{S}(z) \equiv \frac{1}{g_{\mathrm{st}}{}^2} W(z) - \sum_j [2 \ln \vartheta_1(z - \lambda_j) - \ln \vartheta_1(z - \lambda_j + \alpha) - \ln \vartheta_1(z - \lambda_j - \alpha)].$$

is the effective action of a probe eigenvalue placed at z. As usual, the resolvent $R(z)$ is defined to be the force on a probe eigenvalue due to all the other eigenvalues and is given by

$$R(z) = g_{\mathrm{st}}{}^2 \sum_j \frac{\vartheta_1'(z - \lambda_j)}{\vartheta_1(z - \lambda_j)}.$$

In the large $\hat{N}$ limit, the eigenvalues condense into cuts on the z-plane and the saddle point equation for the eigenvalues is expressed in terms of the resolvent and $S \equiv g_{st}^2 \hat{N}$ as

$$W'(z) - S\,(2R(z) - R(z+\alpha) - R(z-\alpha)) = 0, \tag{10}$$

whenever z is on a cut. Following [18], if we can express $W(z)$ as

$$W(z) = U(z + \frac{\alpha}{2}) - U(z - \frac{\alpha}{2}), \tag{11}$$

for some possibly quasiperiodic function $U(z)$ on Σ_τ, we can re-express (10) as

$$J(z + \frac{\alpha}{2} \pm i\epsilon) = J(z - \frac{\alpha}{2} \mp i\epsilon), \tag{12}$$

where $J(z)$ is the auxiliary function

$$J(z) = U'(z) + S[R(z + \frac{\alpha}{2}) - R(z - \frac{\alpha}{2})]. \tag{13}$$

We note that only $U'(z)$, not $U(z)$, need be elliptic. This reformulation of the saddle point equation states that $J(z)$, which is already doubly periodic, is discontinuous at the cuts. In fact the saddle point equation implies that $J(z)$ is a function on a genus 2 Riemann surface Σ_2 obtained from the (dual) T^2 of complex structure τ by cutting it along two segments and gluing the cuts to each other. As depicted in figure (2), the top side of the upper cut is glued to the bottom side of the lower cut and vice versa, yielding two new cycles A_2 and B_2.

Given the function $J(z)$ on Σ_2 one can complete the DV approach by writing down the gaugino superpotential as follows. The single cut solution of the matrix model corresponds in the physical gauge

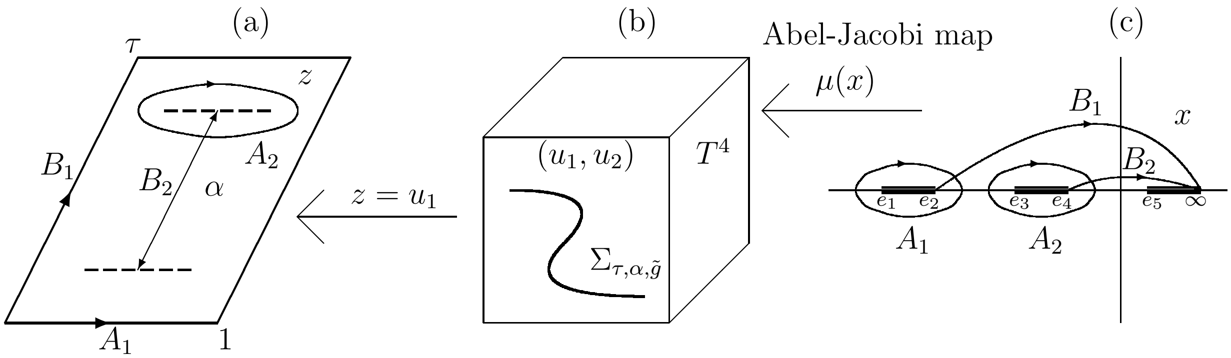

Figure 2: The genus-2 Riemann surface Σ_2 can be represented as: (a) T^2 cut and glued along two parallel segments at a distance α from each other; (b) holomorphic curve inside T^4; (c) hyperelliptic curve on the x-plane, with branch points at $e_1, e_2, e_3, e_4, e_5, \infty$.

theory to a classically unbroken $SU(N)$ vacuum which then quantum mechanically confines, generating a mass gap and yielding a gaugino condensate S. In the matrix model, S is given by

$$S = -\frac{1}{2\pi}\int_{A_2} J(z)dz. \tag{14}$$

The matrix model free energy in the planar, large $\hat{N}$ limit is then given by

$$\frac{\partial \mathcal{F}_0}{\partial S} = -i\int_{B_2} J(z)dz, \tag{15}$$

From these expressions, one finally arrives at the DV gaugino superpotential

$$W_{\text{eff}}(S) = N\frac{\partial \mathcal{F}_\prime}{\partial S} - 2\pi i\rho S, \tag{16}$$

where ρ is the effective 3+1D $SU(N)$ bare (complex) gauge coupling. To obtain a quantitative check of the DV approach for the compactified 6D theory, one could calculate the value of the superpotential (16) at its minimum and compare to the Seiberg-Witten theory or integrable systems approaches. In order to do this, it is important to have an explicit construction of the genus 2 matrix model spectral curve and functions on it.

Any genus g Riemann surface Σ_g can be embedded in its Jacobian T^{2g} by the Abel-Jacobi map $\mu : \Sigma_g \to T^{2g}$ (see appendix B for some details). In our case the Jacobian is a T^4, whose complex structure is

determined by the period matrix of Σ_2, which we can always choose to be

$$\left(\begin{array}{cccc} 1 & 0 & \Omega_{11} & \Omega_{12} \\ 0 & 1 & \Omega_{12} & \Omega_{22} \end{array} \right). \tag{17}$$

The columns represent the four 1-cycles A_1, B_1, A_2, B_2 of Σ_2 in that order, and the rows represent the two holomorphic 1-forms ω_1, ω_2. Apriori, the Ω_{ij} are arbitrary, but we will determine them soon. We choose complex coordinates $u = (u_1, u_2)$ on T^4. For the special case of genus 2, the image of Σ_2 under the Abel-Jacobi map μ sits inside the T^4 as the zero locus of a particular theta function which we call ϑ_0 [24]:

$$\mu(\Sigma_2) = \left\{ \vartheta_0(u|\Omega) \equiv \vartheta \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 1/2 \end{bmatrix} (u|\Omega) = 0 \right\}. \tag{18}$$

$\vartheta_0(u)$ has the nice property that it is odd under the $\mathbb{Z}_2$ action $u \to -u$, which immediately implies that $\mu(\Sigma_2)$ enjoys the same $\mathbb{Z}_2$ involution symmetry.

Important constraints on the periods of the B-cycles of Σ_2, namely the 2×2 matrix Ω, can be obtained by relating the T^4 embedding of Σ_2 to its presentation in figure (2a) as a torus on the z-plane with two cuts glued together. This presentation implies that there exists a holomorphic, but quasiperiodic function z on Σ_2 with no poles. Such a function can be constructed as a quasiperiodic function on T^4 restricted to $\mu(\Sigma)$. Let π_i , $i = 1 \ldots 4$, be the columns of the period matrix (17). Then $z(u)$ obeys the periodicities

$$z(u + \pi_1) = z(u) + 1 \tag{19}$$

$$z(u + \pi_2) = z(u) \tag{20}$$

$$z(u + \pi_3) = z(u) + \tau \tag{21}$$

$$z(u + \pi_4) = z(u) + \alpha. \tag{22}$$

The only functions on T^4 that have no poles, even when restricted to $\mu(\Sigma_2)$, are u_1 and u_2. Without loss of generality we choose the function z to be u_1. Identifying z with u_1 implies that we must have $\Omega_{11} = \tau$, and $\Omega_{12} = \alpha$ in order to satisfy the periodicities above. Hence our matrix model spectral curve has only one undetermined complex structure modulus Ω_{22}. In the matrix model solution this is related to the size of the cut, while in the physical gauge theory, it is related to the gaugino condensate S. Only after minimizing the DV superpotential (16) will we be able to determine Ω_{22}. Indeed we shall find that $\Omega_{22} = \frac{\rho}{N}$.

Having pinned down somewhat the matrix model spectral curve, we now turn to the description of the function $J(z)$. It may help to retrace the steps leading to the definition (13) of $J(z)$ with a specific $\mathcal{N} = 1$

deformation W in mind as an example. For convenience, we choose the deformation

$$W(z) = \zeta(z+\alpha) - \zeta(z-\alpha). \tag{23}$$

This is an elliptic function on the compactification torus Σ_τ, the space on which the eigenvalues of the matrix model live. The eigenvalues like to sit near the critical points of W given by

$$\partial_z W = -\wp(z+\alpha) + \wp(z-\alpha) = 0. \tag{24}$$

This equation is satisfied for $z=0$ because $\wp(z)$ is an even function so we will look for a single cut solution where the eigenvalues condense around $z=0$. Next we need to find a function $U(z)$ satisfying (11). Given our choice of W, (11) is trivially satisfied by $U(z) = \zeta(z)$. $U(z)$ is only a quasiperiodic function on Σ_τ, but its derivative $U'(z) = -\wp(z)$ is nevertheless fully periodic. So finally $J(z)$ in (13) is given by

$$J(z) = -\wp(z) + S[R(z+\frac{\alpha}{2}) - R(z-\frac{\alpha}{2})],$$

and via the saddle point equation (12) is a function on the genus 2 spectral curve Σ_2. The function is determined by its pole structure, specified by $U'(z)$ and its symmetries. We note that $J(z)$ has a single pole on Σ_2 of order 2.

In order to calculate the DV superpotential (16), we must fix the one form $J(z)dz$ and calculate its periods on Σ_2. In doing this it can be helpful to use a hyperelliptic representation of Σ_2. Any genus-2 Riemann surface can be represented as a hyperelliptic curve, i.e. a double cover of the complex plane (see figure 2):

$$y^2 = \prod_{i=1}^{5}(x-e_i). \tag{25}$$

A basis for the holomorphic 1-forms on Σ_2 is given by

$$\omega_1 \equiv \frac{dx}{y}, \qquad \omega_2 \equiv \frac{xdx}{y}.$$

At $x=\infty$ a good coordinate is $w \equiv x^{-\frac{1}{2}}$ so that $x = 1/w^2$ and $y \sim 1/w^5$ and the above forms are nonsingular at $w=0$.

To proceed, we need a meromorphic 1-form with a single singularity of degree 2. The form

$$J = \frac{x^2 dx}{y}$$

has a double pole at $x=y=\infty$ and is the 1-form we are looking for.

We can also express J in terms of the embedding $i : \Sigma_2 \hookrightarrow T^4$. To do this we identify

$$\omega_1 = \frac{dx}{y} = i^* du_1, \qquad \omega_2 = \frac{xdx}{y} = i^* du_2.$$

On Σ_2 we have

$$\partial_1\vartheta_0 du_1 + \partial_2\vartheta_0 du_2 = 0 \Longrightarrow \left.\frac{du_2}{du_1}\right|_{\Sigma_2} = -\left.\frac{\partial_1\vartheta_0}{\partial_2\vartheta_0}\right|_{\Sigma_2},$$

We can therefore set

$$x = \left.\frac{\omega_2}{\omega_1}\right|_{\Sigma_2} = -\left.\frac{\partial_1\vartheta_0}{\partial_2\vartheta_0}\right|_{\Sigma_2}$$

and

$$J = -\left.\frac{\partial_1\vartheta_0}{\partial_2\vartheta_0}du_1\right|_{\Sigma_2}.$$

It can be daunting to evaluate the period integrals of $J(z)$ directly, but as explained in [25], one can nevertheless find the moduli of the matrix model spectral curve at a critical point of the DV superpotential indirectly. The crucial insight is to realize that the 1-form

$$\beta = -\frac{1}{2\pi}\frac{\partial}{\partial S}J(z)dz \tag{26}$$

is actually a holomorphic 1-form. This is because the only singularity in $J(z)$, coming solely from $U'(z)$, is manifestly independent of the modulus S associated to the cut. We know the integral of this one form over the A_1 cycle in figure (2) is 0, since the integral $\int_{A_1} J(z)dz$ is identically 0 independent of S. Therefore β, in the T^4 language, cannot be du_1 and hence must be du_2. Now the extremum condition for the gaugino superpotential (16) is

$$N\frac{\partial^2\mathcal{F}_0}{\partial S^2} = 2\pi i\rho. \tag{27}$$

Using the relation (26) with $\beta = du_2$, we can rewrite

$$\frac{\partial^2\mathcal{F}_0}{\partial S^2} = 2\pi i\int_{B_2} du_2 = 2\pi i\,\Omega_{22}. \tag{28}$$

Thus, as promised, at the critical point of the gaugino superpotential, the one undetermined modulus Ω_{22} of the spectral curve is fixed by (27) and (28) to be $\frac{\rho}{N}$. Hence the period matrix (17) for the spectral curve at the critical point becomes

$$\begin{pmatrix} 1 & 0 & \tau & \alpha \\ 0 & 1 & \alpha & \frac{\rho}{N} \end{pmatrix}. \tag{29}$$

This is very similar to the period matrix (41) of the physical T^4 in which the Seiberg-Witten curves of the undeformed $\mathcal{N} = 2$ theory live. Indeed the moral so far of the six-dimensional DV story is that the Jacobian of the genus 2 matrix model spectral curve associated to a one-cut solution, is closely related to the physical T^4 in which the Seiberg-Witten curves of the undeformed $\mathcal{N} = 2$ theory live. We will investigate this relation in more detail in the next section, but for now we show, again following [25], that the evaluation of the DV superpotential at its critical point can be reduced to the evaluation of a residue.

Assembling previous results, we have

$$
\begin{aligned}
W_{eff} &= N\frac{\partial \mathcal{F}_0}{\partial S} - 2\pi i\rho S && (30)\\
&= iN\Big(-\int_{B_2} J(z)dz + \frac{\rho}{N}\int_{A_2} J(z)dz\Big) && (31)\\
&= iN\Big(-\int_{A_2} du_2 \int_{B_2} J(z)dz + \int_{B_2} du_2 \int_{A_2} J(z)dz\Big) && (32)\\
&= \mathrm{Res}_{z\to P}\Big(U(z)u_2\, dz\Big). && (33)
\end{aligned}
$$

In the last line we used a Riemann bilinear relation assuming $U(z)$ has no simple poles. Here P is the location of the higher order pole of $U(z)$. Fortunately, no daunting integrals need be calculated to extract physical information from the matrix model.

4 The DV matrix integrals for $c_1 \neq 0$

We will now study the $\mathcal{N}=1$ compactifications described in subsection (2.2). We take the chiral field multiplet Φ_+ to live in a line bundle $\mathcal{L}$ on T^2 with a nonzero first Chern class $c_1(\mathcal{L}) = k$. The other chiral multiplet Φ_- takes values in $\mathcal{L}^{-1}$. For simplicity, we set $k=1$. To specify the field theoretic action we must pick a connection

$$
A^{(\mathrm{n.d.})} = A^{(\mathrm{n.d.})}{}_z dz + A^{(\mathrm{n.d.})}{}_{\overline{z}} d\overline{z}
$$

for $\mathcal{L}$ on T^2.

The Matrix Model integral (2) becomes

$$
\int [D\Phi_+][D\Phi_-][D\mathcal{A}_{\overline{z}}] e^{-\int d^2z \,\mathrm{tr}\,\{\Phi_- D_{\overline{z}}\Phi_+ - i\Phi_+[\mathcal{A}_{\overline{z}},\Phi_-]\}} = \int [D\mathcal{A}_{\overline{z}}] \frac{1}{\det(D_{\overline{z}} + i[\mathcal{A}_{\overline{z}},\cdot])}
$$

Here $D_{\overline{z}} = \partial_{\overline{z}} + iA^{(\mathrm{n.d.})}{}_{\overline{z}}$.

To calculate the determinant, we represent T^2 explicitly with variables $0 \leq x_1, x_2 \leq 1$ and

$$
z = x_1 + \tau x_2, \qquad \overline{z} = x_1 + \overline{\tau} x_2, \qquad D_{\overline{z}} = \frac{1}{2i\tau_2}(\tau D_1 - D_2),
$$

where

$$
D_1 \equiv \partial_1 + iA^{(\mathrm{n.d.})}{}_1, \qquad D_2 \equiv \partial_2 + iA^{(\mathrm{n.d.})}{}_2, \qquad \tau \equiv \tau_1 + i\tau_2.
$$

We expand

$$
\Phi_+(x_1,x_2) = \sum_{q=0}^{k-1}\sum_{n=-\infty}^{\infty} e^{-2\pi i(nk+q)x_1}\hat{\Phi}_+^{(q)}(x_2-n) \xrightarrow{k=1} \sum_{n=-\infty}^{\infty} e^{-2\pi i n x_1}\hat{\Phi}_+(x_2-n) \tag{34}
$$

This expansion obeys the boundary conditions for the line bundle $\mathcal{L}$

$$\Phi_+(x_1+1,x_2)=\Phi_+(x_1,x_2), \qquad \Phi_+(x_1,x_2+1)=e^{-2\pi ikx_1}\Phi_+(x_1,x_2).$$

The whole information present in the two-dimensional field Φ_+ is thus encoded in the k one-dimensional fields $\hat{\Phi}^{(q)}$ ($q=0\ldots k-1$). We take the gauge field to be

$$A^{(\mathrm{n.d.})}=2\pi kx_2dx_1 \stackrel{k=1}{\longrightarrow} 2\pi x_2dx_1$$

We will intially restrict to $k=1$ completely; the generalization to positive k is straightforward. It is convenient to define a Heisenberg algebra

$$\hat{a}\equiv\frac{1}{\sqrt{4\pi\tau_2}}(i\partial_2+2\pi\tau x_2), \qquad \hat{a}^\dagger=\frac{1}{\sqrt{4\pi\tau_2}}(i\partial_2+2\pi\overline{\tau}x_2), \qquad [\hat{a},\hat{a}^\dagger]=1.$$

Then

$$D_{\overline{z}}\Phi_+(x_1,x_2)=\sqrt{\frac{\pi}{\tau_2}}\sum_{n=-\infty}^{\infty}e^{2\pi inx_1}[\hat{a}\hat{\Phi}_+](x_2-n)$$

Thus, formally,

$$\det(D_{\overline{z}}+i[\mathcal{A}_{\overline{z}},\cdot])=\prod_{i\neq j}\det\left[(\lambda_i-\lambda_j)\hat{I}+\sqrt{\frac{\pi}{\tau_2}}\hat{a}\right] \tag{35}$$

Here $\hat{I}$ is the identity operator on the representation of the Heisenberg algebra.

Note that since the periodicities of A are

$$A_{\overline{z}}\cong A_{\overline{z}}+\frac{\pi}{\tau_2}\cong A_{\overline{z}}+\frac{\pi\tau}{\tau_2}, \tag{36}$$

the eigenvalues of $A\overline{z}$ take values in a torus with similar periodicities. Then, including the Jacobian, the Matrix model integral (2) becomes

$$\mathcal{Z}=\int d\lambda_1\ldots d\lambda_{\hat{N}}\prod_{i\neq j}\frac{\prod_{m,n}(\lambda_i-\lambda_j+m\frac{\pi}{\tau_2}+n\frac{\pi\tau}{\tau_2})}{\det\left[(\lambda_i-\lambda_j)\hat{I}+\sqrt{\frac{\pi}{\tau_2}}\hat{a}\right]} \tag{37}$$

As it stands, the denominator of (37) does not make sense. The term containing $\hat{a}$ can seemingly be dropped since it does not affect the determinant (being an upper triangular matrix in the harmonic operator representation).

We would like to propose the following regularization of (37). Define the basis of coherent states:

$$|\zeta\rangle=e^{\hat{a}^\dagger\zeta-\frac{1}{2}|\zeta|^2}|0\rangle,$$

where $|0\rangle$ is the ground state of the Heisenberg algebra ($\hat{a}|0\rangle=0$). Then, using

$$\mathrm{tr}\,\mathcal{O}=\frac{1}{\pi}\int d^2\zeta\langle\zeta|\mathcal{O}|\zeta\rangle,$$

we get

$$\log\det\left[(\lambda_i-\lambda_j)\hat{I}+\sqrt{\frac{\pi}{\tau_2}}\hat{a}\right] = \mathrm{tr}\,\log\left[(\lambda_i-\lambda_j)\hat{I}+\sqrt{\frac{\pi}{\tau_2}}\hat{a}\right]$$
$$= \frac{1}{\pi}\int d^2\zeta\log\left(\lambda_i-\lambda_j+\sqrt{\frac{\pi}{\tau_2}}\zeta\right)$$

We redefine $\zeta\to\sqrt{\frac{\pi}{\tau_2}}$. At this point we would like to replace the $d^2\zeta$ integration over the whole $\mathbb{C}$ plane by a sum over integer pairs (n,m) and an integral over a fundamental domain that is a T^2 with sides $\frac{\pi}{\tau_2}$ and $\frac{\pi\tau}{\tau_2}$.

$$\zeta = n\frac{\pi}{\tau_2}+m\frac{\pi\tau}{\tau_2}+\eta,\qquad \text{with}\,\eta\in T^2.$$

and for any expression $F(\zeta)$ we replace

$$\int_{\mathbb{C}}F(\zeta)d^2\zeta\longrightarrow\sum_{n,m}\int_{T^2}d^2\eta F(\eta+m\frac{\pi}{\tau_2}+n\frac{\pi\tau}{\tau_2}).$$

We obtain

$$\int d^2\zeta\log\left(\lambda_i-\lambda_j+\zeta\right)=\sum_{n,m}\int_{T^2}d^2\eta\log\left(\lambda_i-\lambda_j+m\frac{\pi}{\tau_2}+n\frac{\pi\tau}{\tau_2}+\eta\right)$$

The motivation behind this regularization becomes apparent if recall that large values of ζ can be interpreted classically. For example if $\zeta\approx m'\frac{pi}{\tau_2}+n'\frac{\pi\tau}{\tau_2}$ then the corresponding wave-function $\hat{\Phi}^{(q)}(x_2)$ is localized near $x_2\approx m'$. The dominant term in (34) will then have $n\approx n'$. For large values of (m',n') the wave function will therefore behave like $\exp(im'x_1+in'x_2)$. Therefore, it makes sense to rewrite the denominator of (37) so that terms with ζ's near $m\frac{\pi}{\tau_2}+n\frac{\pi\tau}{\tau_2}$ should combine with the term $(\lambda_i-\lambda_j+m\frac{\pi}{\tau_2}+n\frac{\pi\tau}{\tau_2})$ in such a way that the total product will be finite.

Using this, and noting that $\mathrm{vol}(T^2)=\int_{T^2}d^2\eta=\frac{\pi^2}{\tau_2}$, we can find

$$\prod_{i\neq j}\frac{\prod_{m,n}(\lambda_i-\lambda_j+m\frac{\pi}{\tau_2}+n\frac{\pi\tau}{\tau_2})}{\det\left[(\lambda_i-\lambda_j)\hat{I}+\sqrt{\frac{\pi}{\tau_2}}\hat{a}\right]}=$$
$$\exp\left[\frac{\tau_2}{\pi^2}\int_{T^2}d^2\eta\sum_{i<j}\log\frac{\vartheta_1^2(\lambda_i-\lambda_j)}{\vartheta_1(\lambda_i-\lambda_j-\eta)\vartheta_1(\lambda_i-\lambda_j+\eta)}\right] \tag{38}$$

Inserting the α-twists from our previous case, and expressing the Kähler modulus of the eigenvalue torus as $\beta=\pi^2/\tau_2$, we obtain

$$\mathcal{Z}=\int d\lambda_1\dots d\lambda_{\hat{N}}\prod_{i<j}\exp\left[\frac{1}{\beta}\int_{T^2}d^2\eta\log\frac{\vartheta_1^2(\lambda_i-\lambda_j)}{\vartheta_1(\lambda_i-\lambda_j-\alpha-\eta)\vartheta_1(\lambda_i-\lambda_j+\alpha+\eta)}\right]. \tag{39}$$

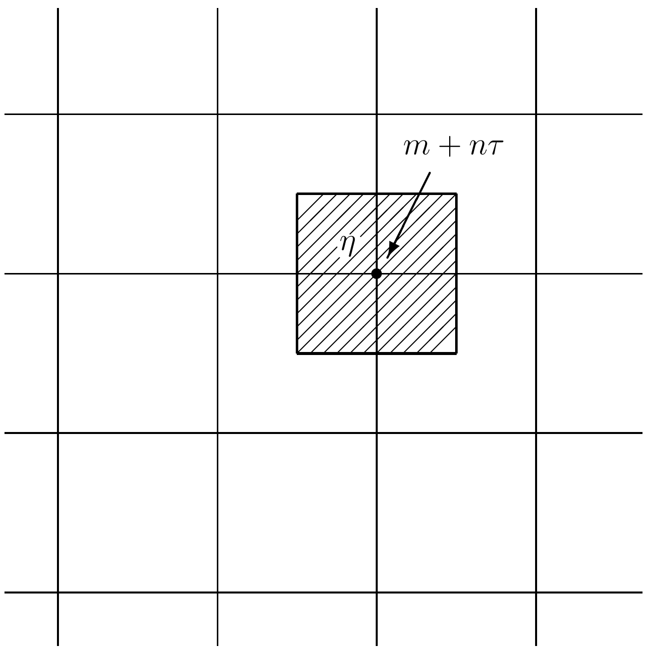

Figure 3: The ill-defined expression (37) is regularized in (39) by splitting the denominator into little pieces, each being an integral over a T^2 centered around the lattice point $m + n\tau$.

Note that this matrix integral is similar to the usual one (8) except for the fact that an $\mathcal{N} = 1$ term is not added by hand. Instead the effects of the SUSY breaking nontrivial R-symmetry Chern class manifests itself in the matrix integral as a smeared background charge in which the eigenvalues float. It would be interesting to analyze this integral further, but for now we turn to an analysis of these compactified six-dimensional theories from the Seiberg-Witten and integrable systems viewpoint.

5 The Integrable Systems Approach

A useful point of view in Seiberg-Witten theory is to view the total space $\mathcal{M}$ of Jacobians of the Seiberg-Witten curve as the phase space of an algebraic integrable system [26]. More specifically, $\mathcal{M}$ is a fibration of abelian varieties (complex projective tori) with base space $\mathcal{M}_{\mathrm{SW}}$, the moduli space of Seiberg-Witten curves. If Σ is a Seiberg-Witten curve with some particular moduli $p \in \mathcal{M}_{\mathrm{SW}}$, the fiber $\pi^{-1}(p)$ of the projection $\pi : \mathcal{M} \to \mathcal{M}_{\mathrm{SW}}$ is simply the Jacobian $\mathcal{J}(\Sigma)$ of Σ. If Σ is a curve of genus g then $\mathcal{J}(\Sigma)$ is a complex g dimensional abelian variety. As is well known, the complex structure of $\mathcal{J}(\Sigma)$ controls the holomorphic couplings of the g $U(1)$ photons present at generic points in the moduli space of vacua of the $4D$ theory.

The utility of this viewpoint is made manifest upon compactifying the 4D theory on a circle S^1 of

radius R to obtain a 3D theory with 8 supercharges. While the moduli space of vacua of the 4D theory is $\mathcal{M}_{\mathrm{SW}}$, the moduli space of vacua of the 3D theory is enlarged to $\mathcal{M}$ [27]. The extra complex scalars in 3D come from dimensionally reducing the 4D vector, and are comprised of Wilson loops and dual photons. The holomorphic couplings of the 4D $U(1)$ photons reduce to kinetic terms for these 3D scalars, and hence the Jacobian of the Seiberg-Witten curve becomes part of the moduli space of the 3D theory. This moduli space of vacua is hyperkahler, but its complex structure can be chosen to be independent of R [27].

After softly breaking the $4D$ $\mathcal{N}=2$ theory by an $\mathcal{N}=1$ superpotential W, the compactified $3D$ theory also develops a superpotential $\mathcal{W}$. The $3D$ superpotential $\mathcal{W}$ is a meromorphic function on $\mathcal{M}$, which as noted above is the phase space of an integrable system. It turns out that $\mathcal{W}$ is generically some combination of the $\frac{1}{2}\mathrm{Dim}_{\mathbb{C}}\mathcal{M}$ Poisson commuting Hamiltonians associated with the integrable system on $\mathcal{M}$. The relationship between the $\mathcal{N}=1$ deformation W and the $3D$ superpotential $\mathcal{W}$ can be made explicit for supersymmetric gauge theories when the associated integrable system has a Lax matrix formulation [30].

After such a relationship has been identified, one can ask for the supersymmetric vacua of the $3D$ theory, given by extrema of $\mathcal{W}$ on $\mathcal{M}$. In the integrable systems language, these extrema are nothing more than fixed points of the Hamiltonian flow on $\mathcal{M}$ generated by $\mathcal{W}$. In other words there is a correspondence between equilibrium configurations of the integrable system with respect to the Hamiltonian $\mathcal{W}$ and supersymmetric vacua surviving the $\mathcal{N}=1$ deformation W [25].

Furthermore, as explained in [25], the integrable systems approach also sheds light on the alternate Dijkgraaf-Vafa approach to calculating the vacuum structure of the $\mathcal{N}=1$ deformed theory. Briefly, at an equilibrium point $p\in\mathcal{M}$ of the Hamiltonian $\mathcal{W}$, the Seiberg-Witten curve associated to that point can be shown to degenerate to a curve of lower genus. At massive vacua, the curve maximally degenerates. In the Dijkgraaf-Vafa approach, minimization of the gluino superpotential amounts to a condition on the matrix model spectral curve that relates its moduli to that of the maximally degenerated Seiberg-Witten curve. Again see [25] and references therein for more details.

These beautiful relationships between massive vacua of $\mathcal{N}=1$ supersymmetric gauge theories, equilibrium points of integrable systems, maximally degenerated Seiberg-Witten curves, and matrix model spectral curves are expected to also hold for the $6D$ twisted, compactified gauge theories considered in this paper. This is plausible because the $4D$ and $5D$ cases in which evidence for such relationships have been accumulated [28, 29, 30] are all continuously connected to the $6D$ case via degenerations. In addition, integrable systems for other $5D$ and $6D$ theories have been found in [38, 37]. A critical step towards obtaining these results in $6D$ is an understanding of the underlying integrable system governing the theory,

a task to which we now turn.

In order to understand the integrable system, we first review and reformulate what is known about the Seiberg-Witten curves of the 6D theory, which were found in [31, 20] as spectral curves of noncommutative instantons on T^4 via string duality arguments. For $SU(N)$ gauge theory the Seiberg-Witten curves are divisors, or zero loci of holomorphic sections, of a line bundle $\mathcal{L}$ over T^4. If we choose real coordinates $x_1, \ldots, x_4$ on T^4 such that $0 \leq x_i \leq 1$ and periodic boundary conditions given by $x_i \sim x_i + 1$, then the Chern class ψ of $\mathcal{L}$ is given by

$$\psi = dx_1 \wedge dx_3 + N\, dx_2 \wedge dx_4. \tag{40}$$

Choosing holomorphic coordinates u_1 and u_2, the period matrix of T^4 (integrals of du_1, du_2 over the 4 1-cycles of T^4) is expressed in terms of the compactification data as

$$\left(\begin{array}{cccc} N & 0 & \tau & \alpha \\ 0 & 1 & \alpha & \rho \end{array} \right), \tag{41}$$

where τ is the complex structure of the T^2 compactification, α is the twist parameter, and ρ is the $4D$ effective complex gauge coupling [31, 20]. $\mathcal{L}$ has precisely N sections which we call ϑ_i, for $i = 0, \ldots, N-1$. In terms of the theta functions defined in appendix B, we can choose ϑ_i to be

$$\vartheta_i \equiv \vartheta \left[\begin{array}{c} i/N \\ 0 \\ 0 \\ 0 \end{array} \right] (u|\Omega).$$

With this data, the Seiberg-Witten curves are written explicitly as zero loci of sections of $\mathcal{L}$ given by the equation

$$\sum_{i=1}^{N-1} a_i \vartheta_i = 0. \tag{42}$$

Each of these Seiberg-Witten curves have genus $N+1$, as can be derived from the adjunction formula, using the fact that divisors of $\mathcal{L}$ are Poincaré dual to the Chern class ψ. The moduli space of Seiberg-Witten curves $\mathcal{M}_{\mathrm{SW}}$ is given by the space of complex coefficients $a_0, \ldots, a_{N-1}$ modulo overall rescalings and hence is $\mathbb{P}^{N-1}$.

The complex $2N-2$ dimensional phase space $\mathcal{M}$ of the underlying integrable system should be the total space of a family of complex $N-1$ dimensional abelian varieties fibered over $\mathcal{M}_{\mathrm{SW}} = \mathbb{P}^{N-1}$. If $p \in \mathbb{P}^{N-1}$, and Σ is the Seiberg-Witten curve associated to p, then the $N-1$ dimensional abelian variety sitting above p in $\mathcal{M}$ can be obtained from the part of $\mathcal{J}(\Sigma)$ that does not come from T^4. More precisely, the two holomorphic one forms du_1, du_2 on T^4 pullback to holomorphic one forms ω_1, ω_2 on Σ via its embedding map into T^4. These one forms in turn span a complex 2-dimensional subspace of the $N+1$

dimensional abelian variety $\mathcal{J}(\Sigma)$. Modding out by this subspace yields the $N-1$ dimensional variety we seek.

One can think of coordinates on $\mathbb{P}^{N-1}$ and coordinates on the $N-1$ dimensional abelian varieties as the action and angle variables respectively of some system of particles. However, in order to gain intuition for the above rather abstractly presented integrable system, it would be useful to actually have a particle-like interpretation of its degrees of freedom, and an explicit formula for its $N-1$ Poisson commuting Hamiltonians. This would be a likely first step in finding equilibrium configurations and connecting to the DV approach. The technique we use to obtain this particle-like interpretation is the separation of variables [32]. The geometry behind this approach was explained in [33] which we now briefly review.

Given a surface S which we can take generally to be a $K3$ manifold, one can define an integrable system $\mathcal{M}$ whose Poisson commuting action variables parameterize the moduli space of holomorphic curves Σ dual to a fixed cohomology class ψ and hence of fixed genus g. The angle variables will parameterize the moduli space of line bundles $\mathcal{L}$ of degree g on Σ. This moduli space for a given Σ is just $\mathcal{J}(\Sigma)$. A key observation is that generically one can map the total space $\mathcal{M}$ of this integrable system to $\mathrm{Symm}^g S$ as follows. A generic section of $\mathcal{L}$ has g zeros. Simply map the point $(\Sigma, \mathcal{L})$ of $\mathcal{M}$ to the point in $\mathrm{Symm}^g S$ represented by these g zeroes. The inverse map can also be constructed. Given g points in S, there is generically a unique curve Σ dual to the cohomology class ψ passing through those points, with a line bundle $\mathcal{L}$ having those g points as a divisor. Furthermore both these maps are symplectomorphisms, where the symplectic structure on $\mathrm{Symm}^g K3$ is the natural one induced by that on $K3$. Physically, the curve Σ along with line bundle $\mathcal{L}$ can be thought of as a bound state of $D2$ branes in S, and the map to g points, or $D0$ branes is a T-duality map, or at the level of sheaves a Fourier-Mukai transform. The system of g points on S constitute the separated variables of the integrable system $\mathcal{M}$.

Returning to our case, we have $S = T^4$ but we do not have every curve Poincaré dual to ψ in (40) at our disposal. We have only those curves that are divisors of the fixed line bundle $\mathcal{L}$ on T^4. Generically a translate in T^4 of such a divisor is no longer a divisor of $\mathcal{L}$. Motivated by the separation of variables technique, we would like to map our integrable system $\mathcal{M}$ to a system of points on T^4 [1]. Physically, we require a $2N-2$ dimensional phase space, indicating that our integrable system is really a collection of $N-1$ identical particles moving on T^4. We now check that one can construct the inverse map: given $N-1$ points there is a unique divisor Σ of $\mathcal{L}$ going through those $N-1$ points.

To see this, it is useful to consider the canonical embedding $\iota : T^4 \to \mathbb{P}^{N-1}$ using the space of sections of $\mathcal{L}$. Given a point $x \in T^4$, $\iota(x)$ is a point in $\mathbb{P}^{N-1}$ specified by the homogeneous coordinates

[1] This point was independently noted in [22].

$[\vartheta_0(x), \ldots, \vartheta_{N-1}(x)]$. As long as there is no $x \in T^4$ for which $\vartheta_i(x) = 0 \ \forall \ i$ this map will be well defined. This consistency condition will be met for N sufficiently large. The utility of the canonical embedding ι is that the space of Seiberg-Witten curves can now be viewed as the space of codimension 1 hyperplanes in $\mathbb{P}^{N-1}$. Any such hyperplane slices the embedding of T^4 in P^{N-1} in a Seiberg-Witten curve Σ. Furthermore, any $N-1$ points in general position in P^{N-1} trivially yield a unique hyperplane through those points. If in addition these points are on the image of T^4, this hyperplane carves out the unique Seiberg-Witten curve in T^4 going through those points. As these $N-1$ points move around on Σ, they also parameterize the $N-1$ angle variables of the system.

We are now in a position to explicitly write down the $N-1$ Poisson commuting Hamiltonians of our integrable system as a function of the separated variables consisting of the $N-1$ points $x_1, \ldots, x_{N-1}$ in T^4. We know from above that given generic $x_1, \ldots, x_{N-1}$, we can determine $a_0, \ldots a_{N-1}$ uniquely by the condition that $x_1, \ldots, x_{N-1}$ lie on the same curve, specified by equation (42). Furthermore we know the functions $\frac{a_i}{a_j}$ for $0 \leq i \leq j \leq N-1$ on $\mathbb{P}^{N-1}$ all Poisson commute. We could for example choose an algebraically independent set of $N-1$ Hamiltonians $H_i \equiv \frac{a_i}{a_0}$, $i = 1 \ldots N-1$, which we wish to find as functions of $x_1, \ldots, x_{N-1}$. We do this by solving the relation between $x_1, \ldots, x_{N-1}$ and $a_0, \ldots a_{N-1}$ given by

$$\begin{pmatrix} \vartheta_0(x_1) & \ldots & \vartheta_{N-1}(x_1) \\ \vdots & \ddots & \vdots \\ \vartheta_0(x_{N-1}) & \ldots & \vartheta_{N-1}(x_{N-1}) \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_{N-1} \end{pmatrix} = 0. \tag{43}$$

Let Θ be the $(N-1) \times N$ matrix appearing in (43). Let $\Theta[i]$ be the $(N-1) \times (N-1)$ matrix obtained from Θ by deleting the ith column. Then the reader may check that solving for the a_i in terms of the x_i in (43) yields the desired Hamiltonians

$$\frac{a_i}{a_j} = (-1)^{i+j} \frac{\mathrm{Det}\, \Theta[i]}{\mathrm{Det}\, \Theta[j]}. \tag{44}$$

These Hamiltonians are manifestly symmetric in $x_1, \ldots, x_{N-1}$ and all Poisson commute with respect to the natural symplectic structure on $\mathrm{Symm}^{N-1} T^4$ given by

$$\omega = \sum_{i=1}^{N-1} du_1^{(i)} \wedge du_2^{(i)}. \tag{45}$$

It is instructive to consider the physics of the simplest case of $SU(2)$ gauge theory, where N is now 2. $\mathcal{M}_{\mathrm{SW}}$ is just $\mathbb{P}^1$ with homogenous coordinates $[a_0, a_1]$. The point $[a_0, a_1] \in \mathbb{P}^1$ corresponds to the genus 3 curve $a_0\vartheta_0 + a_1\vartheta_1 = 0$. The phase space $\mathcal{M}$ can be viewed as a T^2 fibration over $\mathbb{P}^1$, or alternatively, away from nongeneric points, as a T^4 with symplectic structure ω in (45). The Hamiltonian of the integrable

system is $H = \frac{a_1}{a_0}$, or in terms of the single separated variable x, $H = -\frac{\vartheta_0(x)}{\vartheta_1(x)}$. The Hamiltonian vector field χ_H is given as usual by the relation $\omega(\chi_H, \centerdot) = \partial H$. It can be easily seen that the equilibrium condition on $x = (u_1, u_2)$ with respect to this Hamiltonian flow χ_H is equivalent to the two equations

$$\vartheta_1(x)\partial_{u_i}\vartheta_0(x) - \vartheta_0(x)\partial_{u_i}\vartheta_1(x) = 0 \qquad i = 1, 2. \tag{46}$$

These can be solved in two qualitatively distinct cases. One is when $\vartheta_0(x) = \vartheta_1(x) = 0$. This case actually occurs at 4 points on T^4 as can be calculated from intersection theory. The second case is nicely interpreted in terms of the unique Seiberg-Witten curve that goes through x. In terms of the moduli of this curve, $[a_0, a_1]$, the equilibrium equations (46) can be rewritten as

$$a_0\partial_{u_i}\vartheta_0(x) + a_1\partial_{u_i}\vartheta_1(x) = 0 \qquad i = 1, 2. \tag{47}$$

However these are simply the conditions that the curve $a_0\vartheta_0 + a_1\vartheta_1$ is singular. Thus we recover very easily the observations in lower dimensions that equilibrium conditions on the phase space of the integrable system are equivalent to degeneration conditions on the Seiberg-Witten curve, at least for the case of $N = 2$.

The proof for larger N is suggested by the above technique. Recall the equation defining the Seiberg-Witten curve:

$$\sum_{i=0}^{N-1} a_i\vartheta_i(x_j) = 0 \tag{48}$$

Now, as stated, this equation only holds for the specific x_j defining the curve. However, as we did above, one can solve for the a_i in terms of the x_j. Defining $\vec{x} = (x_1, \ldots, x_{N-1})$, we see that the equation

$$\sum_{i=0}^{N-1} a_i(\vec{x})\vartheta_i(x_j) = 0 \tag{49}$$

holds for any choice of x_j, or more generally for any choice of $\vec{x}$. Thus, we may take the derivative. Letting $x_j = (u_1^j, u_2^j)$, we see

$$\sum_{i=0}^{N-1} \left[\left(\partial_{u_k^j} a_i(\vec{x}) \right) \vartheta_i(x_j) + a_i(\vec{x})\partial_{u_k}\vartheta_i(x_j) \right] = 0; \quad k = 1, 2; \quad j = 1, \ldots, N-1. \tag{50}$$

The latter term is exactly that which appears in the condition for the Seiberg-Witten curve to be degenerate. If we show the first term is 0, then it follows that the curve degenerates. To show this, use the stationary-point condition for $H_i = a_i/a_0$.

$$0 = \partial_{u_j^k} H_i = \frac{a_0\partial_{u_j^k} a_i - a_i\partial_{u_j^k} a_0}{a_0^2} \tag{51}$$

Now, assuming $a_0 \neq 0$ (otherwise we should have chosen a different basis for our Hamiltonians), this is equivalent to

$$\partial_{u_j^k} a_i = \frac{a_i}{a_0} \partial_{u_j^k} a_0. \tag{52}$$

Armed with this result, we can rearrange the first term above:

$$\begin{aligned} \sum_{i=0}^{N-1} \left(\partial_{u_k^j} a_i(\vec{x}) \right) \vartheta_i(x_j) &= \sum_{i=0}^{N-1} \left(\frac{a_i}{a_0} \partial_{u_k^j} a_0(\vec{x}) \right) \vartheta_i(x_j) & (53) \\ &= \frac{1}{a_0} \partial_{u_k^j} a_0 \sum_{i=0}^{N-1} a_i(\vec{x}) \vartheta_i(x_j). & (54) \end{aligned}$$

But this last sum is 0 because x_j is on the curve. Thus the second term in Eq. (50) is similarly 0, and since this is true for all $\partial_{u_j^k}$, the curve degenerates.

The moral of the story is that if a configuration of points $x_1, \ldots, x_{N-1}$ is at equilibrium with respect to all $N-1$ Hamiltonians, then the Seiberg-Witten curve that goes through these points is singular at each and every one of them. One can view this type of degeneration as $N-1$ pinched cycles. Each time a cycle pinches the genus is reduced by one. Hence the generic genus $N+1$ Seiberg-Witten curve, at such special equilibrium points, has $N-1$ pinched cycles and thus degenerates to genus 2. This degeneration signals that we are in a massive vacuum with no unbroken $U(1)$'s. Furthermore this degenerated curve should be related to the genus two spectral curve coming from the matrix model.

6 A Dual set of Hamiltonians

Here, inspired by [34, 35], we conjecture the existence of an alternate set of Poisson commuting Hamiltonians for the integrable system underlying the six dimensional $SU(N)$ gauge theory which could shed some light on the physical nature of the system. We begin again with the genus $N+1$ Seiberg-Witten curves of our compactified $\mathcal{N}=2$ theory. Their period matrices all exhibit an interesting structure, which is purely a consequence of the fact that they can all be embedded in a T^4.

Let Σ_{N+1} be such a genus $N+1$ curve equipped with an embedding map $\iota : \Sigma_{N+1} \to T^4$, with T^4 having a complex structure specified by the period matrix (41). Let $\alpha_1, \ldots, \alpha_{N+1}, \beta_1, \ldots, \beta_{N+1}$ be a symplectic basis of 1-cycles on Σ_{N+1}, and A_1, B_1, A_2, B_2 be such a basis on T^4. ι induces a map on 1-cycles preserving the intersection product. For convenience, we assume $\alpha_1, \beta_1 \to A_1, B_1$ under ι and all the rest map to A_2, B_2. The argument can be easily modified to accommodate a more general situation. Now Σ_{N+1} has $N+1$ holomorphic one-forms. Choose a basis $\omega_1, \ldots, \omega_{N+1}$ subject to the conditions

$$\begin{aligned} \omega_1 &= \iota^* \tfrac{du_1}{N} & (55) \\ \omega_2 + \cdots + \omega_{N+1} &= \iota^* du_2. & (56) \end{aligned}$$

With this basis of one-forms, one can easily check that their periods over the α cycles yield the identity matrix, and their periods over the β cycles yield the

$$\tilde{T} = \begin{bmatrix} \tau/N & \alpha/N & \cdots & \alpha/N \\ \alpha/N & T_{11} & \cdots & T_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha/N & T_{N1} & \cdots & T_{NN} \end{bmatrix}, \tag{57}$$

where $\sum_i T_{ij} = \sum_j T_{ij} = \rho$. The motivation for putting the period matrix in this form is to allow us to decompose a theta function on $\mathcal{J}(\Sigma_{N+1}) \sim T^{2N+2}$ into a sum over products of theta functions on T^4 and theta functions on T^{2N-2}. Ratios of these latter theta functions should yield an alternate set of Poisson commuting Hamiltonians.

We do this as follows. Consider a theta function on T^{2N+2},

$$\vartheta(\tilde{p}|\tilde{T}) = \sum_{\tilde{n}\in\mathbb{Z}^{N+1}} \exp\left[\pi i \tilde{n}^t \tilde{T} \tilde{n} + 2\pi i \tilde{n}^t \tilde{p}\right].$$

Decompose $\tilde{n} = (m_1, n)$ and $\tilde{p} = (q_1, p)$, where m_1, q_1 are one-dimensional, while n, p are N dimensional. The curious subscripts anticipate the remainder of our decomposition. With these, ϑ can be written as

$$\vartheta(q_1, p|\tilde{T}) = \sum_{n\in\mathbb{Z}^N, m_1\in\mathbb{Z}} \exp\left[\pi i (m_1^2 \tau/N + 2m_1(\sum_i n)\alpha/N + n^t T n) + 2\pi i (m_1 q_1 + n^t p)\right].$$

Following the procedure laid down by [34] and [35], let $T_{ij} = \rho/N + \hat{T}_{ij}$ and $p_i = q_2 + \hat{p}_i$, where

$$\sum_i \hat{p}_i = \sum_i \hat{T}_{ij} = \sum_j \hat{T}_{ij} = 0. \tag{58}$$

Then, combining $q = (q_1, q_2)$, $m = (m_1, m_2)$, and $\Omega = \begin{bmatrix} \tau & \alpha \\ \alpha & \rho \end{bmatrix}$, our function becomes

$$\vartheta(q, \hat{p}|\Omega, \hat{T}) = \sum_{m\in\mathbb{Z}^2} \exp\left[\pi i m^t \frac{\Omega}{N} m + 2\pi i m^t q\right] \sum_{\substack{n\in\mathbb{Z}^N \\ \Sigma_i n_i = m_2}} \exp\left[\pi i n^t \hat{T} n + 2\pi i n^t \hat{p}\right] \tag{59}$$

We express m as $Nm + k$, where $k = 0 \ldots N-1$. This allows us to sum over $\vartheta\begin{bmatrix} \frac{k}{N} \\ 0 \end{bmatrix}$ on T^4 (see Appendix B for definitions).

$$\vartheta(q, \hat{p}|\Omega, \hat{T}) = \sum_{k\in\mathbb{Z}_N^2} \vartheta\begin{bmatrix} \frac{k}{N} \\ 0 \end{bmatrix}(Nq|N\Omega)\, \hat{\vartheta}_{k_2}^{(N-1)}(\hat{p}|\hat{T}), \quad \text{where} \tag{60}$$

$$\hat{\vartheta}_{k_2}^{(N-1)}(\hat{p}|\hat{T}) = \sum_{\substack{n\in\mathbb{Z}^N \\ \Sigma_i n_i = k_2}} \exp\left[\pi i n^t \hat{T} n + 2\pi i n^t \hat{p}\right], \qquad k_2 = 0 \ldots N-1. \tag{61}$$

In the limit where this integrable system degenerates to the elliptic Calogero-Moser system relevant for the $\mathcal{N}=1^*$ theory, ratios of the N functions defined in (61) were shown to Poisson commute in [35]. It is reasonable to conjecture this continues to holds true away from that limit. Moreover, these functions where shown to be dependent only on the coordinates of the Calogero-Moser particles. Hence a deeper understanding of these functions might shed light on a particle/coordinate interpretation of the integrable system relevant for $6D$ gauge theory rather than just a phase space interpretation.

Intuitively, $\hat{\vartheta}^{(N-1)}_{k_2}(\hat{p}|\hat{T})$ in (61) is a function on the part of $\mathcal{J}(\Sigma_{N+1})$ that does not come from T^4, and hence is a function of the angle variables. However $\hat{\vartheta}^{(N-1)}_{k_2}(\hat{p}|\hat{T})$ also depends on the action variables though its dependence on $\hat{T}$. If one thought of this function as a function of the separated variables $x_1, \ldots x_{N-1}$, then unlike the action variables, which depend only on the curve determined by these $N-1$ points, $\hat{\vartheta}^{(N-1)}_{k_2}(\hat{p}|\hat{T})$ also depends on where these points are on the curve. This particular combination of action and angle variables could be interpreted as functions of the "coordinates" of the integrable system.

7 Summary and discussion

In this paper we have made some progress towards exploring the circle of ideas relating matrix models, Seiberg-Witten curves and integrable systems in the context of massive vacua of twisted, compactified 6D gauge theories. We have observed a great deal of unity in the three approaches, each of which ultimately gives its answer in the form of a Riemann surface.

In the matrix model approach the Riemann surface arises microscopically through the condensation of matrix model eigenvalues. Because these eigenvalues live on a compact two torus, the spectral curve is of genus two. Upon extremization of the DV superpotential, the complex structure of the Jacobian of this Riemann surface is related to a physical T^4 in which the Seiberg-Witten curves live.

In the integrable systems approach, the moduli space of vacua upon further compactification is shown to be a system of $N-1$ points living on this same T^4. The relevant data here are a set of Poisson commuting Hamiltonians, and the condition for a massive vacuum is that the configuration of the integrable system be at equilibrium with respect to all Hamiltonians.

This integrable system condition is shown to reproduce the Seiberg-Witten condition for a massive vacuum, namely that the Seiberg-Witten curve degenerates to genus 2, which again is the same as the matrix model spectral curve. As a further confirmation of these ideas it would be interesting to calculate the actual values of the $\mathcal{N}=1$ superpotential in the massive vacua. This would relate residue calculations on the matrix model spectral curve to the values of integrable system Hamiltonians at equilibrium points.

In 4D the Lax-pair formulation of the integrable system provides a nice dictionary between $\mathcal{N}=1$

superpotentials and integrable system Hamiltonians [30]. It would be useful to have this dictionary in six dimensions. One could also easily generalize this paper to quiver models with k hypermultiplets. This would lead to a fairly simple spin generalization of the system of points on T^4, generalizing the spin generalizations found in lower dimensions. Also a particle like interpretation of the 6D integrable system would be useful, and might be related to the ansatz for a dual set of Poisson commuting Hamiltonians in section 6.

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A Meromorphic functions on T^2

Consider a torus T^2 with complex structure τ ($\mathrm{Im}\,\tau > 0$). A theta function ϑ on the torus is a quasi-periodic function, with the following periodicity conditions:

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z+1|\tau) = e^{2\pi i a}\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z|\tau) \tag{62}$$

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z+\tau|\tau) = e^{-\pi i\tau - 2\pi i(z+b)}\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z|\tau) \tag{63}$$

An explicit formula for ϑ is

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z|\tau) = \sum_{n=-\infty}^{\infty} \exp[\pi i(n+a)^2\tau + 2\pi i(n+a)(z+b)]. \tag{64}$$

We can use three methods to construct meromorphic functions on T^2 from theta functions. The first is to form ratios.

$$\frac{\prod_{i=1}^{n}\vartheta\left[\begin{smallmatrix}a_i\\b_i\end{smallmatrix}\right](z|\tau)}{\prod_{i=1}^{n}\vartheta\left[\begin{smallmatrix}a'_i\\b'_i\end{smallmatrix}\right](z|\tau)}$$

is a meromorphic function on T^2 provided $\sum a_i \equiv \sum a'_i, \sum b_i \equiv \sum b'_i \mathrm{mod}\,\mathbb{Z}$. Another method is the derivative of the logarithm of a ratio of theta function.

$$\partial_{z_i}\ln\frac{\vartheta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](z|\tau)}{\vartheta\left[\begin{smallmatrix}a'\\b'\end{smallmatrix}\right](z|\tau)}$$

The last is the second derivative of the logarithm of a theta function.

$$\partial_{z_i}\partial_{z_j}\ln\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z|\tau)$$

Using the transformation properties above, these can be shown to be periodic in both directions.

Another method of constructing meromorphic functions on a T^2 involves the Weierstrass $\wp$-function, and its derivative $\wp'$. We define it as

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda,\lambda\neq 0}\left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right), \tag{65}$$

where $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ is the lattice defining the torus. This is even, (doubly) periodic in Λ, analytic on $\mathbb{C}\backslash\Lambda$, and has a pole of order two at the points on Λ. $\wp$ and $\wp'$ satisfy the differential equation

$$\wp^2 = 4\wp^3 - g_2\wp - g_3, \tag{66}$$

where g_2 and g_3 are constants determined by the lattice Λ (and therefore τ). Note that as $\wp$ is even and doubly periodic, $\wp'$ is odd and doubly periodic. It turns out that any doubly periodic function F can be written as

$$F(z) = R_1(\wp) + \wp' R_2(\wp), \tag{67}$$

with R_1 and R_2 rational functions. Morally one decomposes F into odd and even parts.

Two other Weierstrass functions deserve mention: the Weierstrass σ function and the Weierstrass ζ function, the latter not to be confused with the Riemann ζ function. We define them as

$$\sigma(z) = z \prod_{\lambda\in\Lambda,\lambda\neq 0} \left(1 - \frac{z}{\lambda}\right) \exp\left[\frac{z}{w} + \frac{1}{2}\left(\frac{z}{\lambda}\right)^2\right] \tag{68}$$

$$\zeta(z) = \frac{1}{z} + \sum_{\lambda\in\Lambda,\lambda\neq 0} \left(\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2}\right). \tag{69}$$

ζ has a simple pole with residue 1 at every point in Λ, and is analytic on $\mathbb{C}\backslash\Lambda$. Lastly, we note the relations between these various functions and their periodicity properties.

$$\zeta(z) = \frac{d}{dz}\log\sigma(z) \tag{70}$$

$$\wp(z) = -\frac{d}{dz}\zeta(z) \tag{71}$$

$$\zeta(z+n+m\tau) = \zeta(z) + n\eta_1 + m\eta_2 \tag{72}$$

$$\sigma(z+n+m\tau) = (-1)^{nm+n+m}\sigma(z)\exp\left[(n\eta_1+m\eta_2)(z+\frac{1}{2}(n+m\tau))\right] \tag{73}$$

$$\eta_1\tau - \eta_2 = 2\pi i. \tag{74}$$

B Higher dimensional Theta functions

For a higher dimensional complex torus $\mathbb{C}^g/\Lambda$, where Λ is a lattice of rank $2g$, the analogy of the complex structure τ is a $g\times g$ complex matrix Ω. Ω must be symmetric, and $\mathrm{Im}\,\Omega$ must be positive definite. Then the higher dimensional ϑ functions are defined on $\mathbb{C}^g$ as

$$\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\vec{z}|\Omega) = \sum_{\vec{n}\in\Lambda}\exp[\pi i(\vec{n}+\vec{a})\cdot\Omega\cdot(\vec{n}+\vec{a}) + 2\pi i(\vec{n}+\vec{a})\cdot(\vec{z}+\vec{b})]. \tag{75}$$

Similar to the T^2 case, where we could holomorphically transform the lattice to $\mathbb{Z}+\tau\mathbb{Z}$, in the g-complex dimensional case, we can view the lattice as $\Lambda = \mathbb{Z}^g + \Omega\mathbb{Z}^g$. The periodicity properties are also analogous. Let $\vec{m}\in\mathbb{Z}^g$.

$$\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\vec{z}+\vec{m}|\Omega) = e^{2\pi i\vec{a}\cdot\vec{m}}\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\vec{z}|\Omega), \tag{76}$$

$$\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\vec{z}+\Omega\vec{m}|\Omega) = e^{-\pi i\vec{m}\cdot\Omega\vec{m}-2\pi i\vec{m}\cdot(\vec{z}+\vec{b})}\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\vec{z}|\Omega). \tag{77}$$

We can use the same three methods as before to construct meromorphic functions on $T^{2g} = \mathbb{C}^g/\Lambda$ using theta functions. We repeat them here in the multi-dimensional notation for appendectical completeness.

$$\frac{\prod_{i=1}^{n} \vartheta \left[\begin{smallmatrix} \vec{a}_i \\ \vec{b}_i \end{smallmatrix}\right](\vec{z}|\Omega)}{\prod_{i=1}^{n} \vartheta \left[\begin{smallmatrix} \vec{a}_i{}' \\ \vec{b}_i{}' \end{smallmatrix}\right](\vec{z}|\Omega)} \tag{78}$$

is a meromorphic function on T^{2g} provided $\sum a_i \equiv \sum a_i', \sum b_i \equiv \sum b_i' \bmod \mathbb{Z}^g$.

$$\text{So are} \quad \partial_{z_i} \ln \frac{\vartheta \left[\begin{smallmatrix} \vec{a} \\ \vec{b} \end{smallmatrix}\right](\vec{z}|\Omega)}{\vartheta \left[\begin{smallmatrix} \vec{a}' \\ \vec{b}' \end{smallmatrix}\right](\vec{z}|\Omega)} \quad \text{and} \quad \partial_{z_i}\partial_{z_j} \ln \vartheta \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix}(\vec{z}|\Omega), \tag{79}$$

for any choice of characters. Again, using the above transformation properties, these can be shown to be periodic in all $2g$ directions.

These functions can be used to define functions on genus g Riemannian surfaces. Consider such a Σ_g. There are g holomorphic 1-forms on Σ_g, call them ω_i. Denote the canonical basis of $H_1(\Sigma_g, \mathbb{Z})$ by the $2g$ cycles A_i and B_i, where $A_i \cap A_j = B_i \cap B_j = 0$, and $A_i \cap B_j = \delta_{ij}$. Then we can define the $g \times 2g$ *period matrix* as

$$\begin{bmatrix} \int_{A_1} \omega_1 & \cdots & \int_{A_g} \omega_1 & \int_{B_1} \omega_1 & \cdots & \int_{B_g} \omega_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \int_{A_1} \omega_g & \cdots & \int_{A_g} \omega_g & \int_{B_1} \omega_g & \cdots & \int_{B_g} \omega_g \end{bmatrix}. \tag{80}$$

We can choose the ω_i such that $\int_{A_j} \omega_i = \delta_{ij}$; then the period matrix is in the form $[\mathbb{I}, \Omega]$ for the $g \times g$ identity matrix $\mathbb{I}$ and a $g \times g$ symmetric matrix Ω, where $\mathrm{Im}\ \Omega$ is positive definite. This similarity with the complex structure matrix in the beginning of this appendix, which was cunningly also named Ω, is not coincidental, as we now show.

Now the columns of the period matrix are $2g$ vectors in $\mathbb{C}^g$; these naturally form a lattice Λ and thus induce a torus $T^{2g} = \mathbb{C}^g/\Lambda$. This torus is called the *Jacobian* of Σ_g, often denoted $\mathcal{J}(\Sigma_g)$. What is the relation between these two objects? The answer is given by the Abel-Jacobi map.

Choose a $p_0 \in \Sigma_g$, and consider the function $\mu : \Sigma_g \to \mathbb{C}^g/\Lambda$, under which

$$p \mapsto \left(\int_{p_0}^{p} \omega_1, \ldots, \int_{p_0}^{p} \omega_g \right). \tag{81}$$

Note this is only defined up to Λ, since in choosing a contour from p_0 to p we could go around any combination of the cycles of the torus. In fact, we can generalize this as a map from any degree 0 divisor to $\mathcal{J}$. This is the Abel-Jacobi map, $\mu : \mathrm{Div}^0\Sigma_g \to \mathcal{J}(\Sigma_g)$, where

$$\sum_i (p_i - q_i) \mapsto \left(\sum_i \int_{q_i}^{p_i} \omega_1, \ldots, \sum_i \int_{q_i}^{p_i} \omega_g \right). \tag{82}$$

As Σ_g is one-complex dimensional, and $\mathcal{J}$ is g-complex dimensional, in order to get a surjective map we need to pick g points on Σ_g, say p_i, $i = 1, \ldots, g$. The Jacobi Inversion Theorem states this explicitly: Given any $\lambda \in \mathcal{J}(\Sigma_g)$, there exist g points $p_i \in \Sigma_g$ such that $\mu\left(\sum_i (p_i - p_0)\right) = \lambda$. Moreover, these p_i are generically unique. Finally, Abel's theorem states that if the divisor $\sum(p_i - q_i)$ is a divisor of some meromorphic function, then $\mu\left(\sum(p_i - q_i)\right) = 0$. These two results mean the Abel-Jacobi map is an isomorphism between the moduli space of line bundles of degree 0, $\mathrm{Pic}^0(\Sigma_g)$, and the Jacobian $\mathcal{J}(\Sigma_g)$.

As promised, this allows us to find meromorphic functions on Σ_g. The Abel-Jacobi map gives us an embedding of Σ_g into $\mathcal{J}(\Sigma_g)$. By constructing meromorphic functions on $T^{2g} \cong \mathcal{J}(\Sigma_g)$, we can simply pull them back under the Abel-Jacobi map to get meromorphic functions on Σ_g.

More complete expositions can be found in [36] and [24].

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